§0. Kazhdan–Lusztig conjecture

(proved for $\mathfrak{g}$: finite Beilinson–Bernstein, Brylinski–Kashiwara
affine Kashiwara – Tanisaki

$\mathfrak{g}$: finite dim'l cpx simple Lie algebra
$\mathfrak{b}$: Borel $\supset \mathfrak{h}$: Cartan , $\mathfrak{n}$: nilpotent radical , $\mathfrak{n}_-$: opposite

$\lambda^{f.pt} = \lambda^{hw} + \rho$

$\alpha = \lambda^{f.pt} - w\lambda^{f.pt}$

$= (\lambda^{hw} + \rho) - w(\lambda^{hw} + \rho)$

$w(\lambda^{hw}+\rho) = w\circ\lambda^{hw} + \rho$

$= \lambda^{hw} - w\circ\lambda^{hw}$

$w\circ\lambda^{hw} = w(\lambda^{hw}+\rho) - \rho = w\lambda^{f.pt} - \rho$

Take $\lambda \in \mathfrak{h}^* = \{\lambda : \mathfrak{h} \longrightarrow \mathbb{C}\}$, and regard

$$\begin{tikzcd} \mathfrak{b} \arrow[r, two heads] \arrow[rrr, bend right, "\lambda"'] & \mathfrak{b}/\mathfrak{n} = \mathfrak{h} \arrow[r, "\lambda"] & \mathbb{C} \end{tikzcd}$$

Consider $M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}$: Verma module

$M(\lambda) \ni v_\lambda = 1 \otimes 1$ highest weight vector

- $e_i v_\lambda = 0$
- $h_i v_\lambda = \langle h_i, \lambda \rangle v_\lambda$
- $M(\lambda) \cong U(\mathfrak{n}_-)$ as a vector space

<u>Elementary Fact</u> (1) $M(\lambda) \twoheadrightarrow^{\exists 1} L(\lambda)$: simple quotient

(2) $\mathrm{ch}\, L(\lambda - \rho) = \sum_{w \in W(\lambda)}^{\exists} a_w^\lambda \, \mathrm{ch}\, M(w\lambda - \rho)$, $a_w^\lambda \in \mathbb{Z}$

$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$: Weyl vector

(cf. Weyl character formula)

$W(\lambda)$: certain Weyl group determined by λ (Assume $W(\lambda) = W$ hereafter) dot action $\quad w\cdot\lambda = w(\lambda+\rho)-\rho$

A little more precise :

$\exists\, \mathcal{C}$ subcategory of representations of $\mathfrak{g}$ s.t.

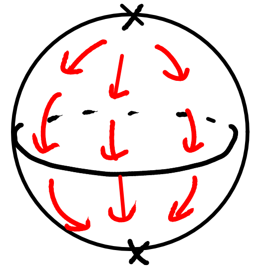
$\{L(w\lambda-\rho) \mid w\in W\}$, $\{M(w\lambda-\rho) \mid w\in W\}$ give two bases of $K(\mathcal{O})$

and $\quad \operatorname{ch} L(w\lambda-\rho) = \sum_{y\geq w} a_{wy} \operatorname{ch} M(y\lambda-\rho)$ $\quad$ (assume λ: dominant)

triangular

<u>Problem</u>. Compute $a_{w,y}\in\mathbb{Z}$

<u>Answer</u>: Singularities of Schubert cells in flag variety.



$\mathcal{B} = G/B$ flag variety $\curvearrowleft T$: maximal torus

$\exists$ natural bijection $\mathcal{B}^T \cong W$: Weyl group

Let $\mathcal{B}_w =$ stable mfd for w $\cong \mathbb{C}^{l(ww_0)}$ $\quad w_0 =$ longest element

$\quad\ \ \mathcal{B}^w =$ unstable mfd for w $\cong \mathbb{C}^{l(w)}$

$\therefore \mathcal{B} = \coprod \mathcal{B}_w = \coprod \mathcal{B}^w$ stratification

<u>KL conjecture</u> $a_{w,y}$ (or its "inverse") is given by dimension of the $^!$stalk of $IC(\mathcal{B}^w)$ at y

computable in a purely combinatorial way (KL polynomial)

Proof by BB, BK, KT : localization of D-modules on $\mathcal{B}$

§1. convolution algebra (extension of Ginzburg theory)

Setting $\quad X$: alg. variety $\curvearrowleft T$: torus

$\mathcal{L}$: semisimple T-equivariant perverse sheaf

- Special class ☆ : $\pi : M \to X$ T-equiv. resolution $\quad \mathcal{L} = \pi_! \mathbb{C}_M$)

Consider $\quad \mathcal{A} := \mathrm{Ext}^*_{D^b_T(\mathrm{cons}X)}(\mathcal{L}, \mathcal{L})$ $\quad$ algebra over $H^*_T(pt) \cong \mathbb{C}[\mathrm{Lie}\,T]$

For ☆ $\quad \cong H^*_T(Z) \qquad Z = M \times_X M$

<u>Known examples</u> $\quad$ 1) (degenerate) affine Hecke algebra $\quad M = T^*\mathcal{B} \longrightarrow X = \mathcal{N}$

(Kazhdan-Lusztig, Ginzburg) $\quad \mathbb{C}^* \times T \circlearrowright$

2) $Y(\mathfrak{g})$ $\qquad M =$ nonsingular quiver variety $\longrightarrow X =$ affinization (Varagnolo)

(Maulik-Okounkov)

not ☆ 3) [Feigin-Finkelberg-Kuznetsov-Mirković]

Zastava space = X^α : partial cptification of $\mathrm{Map}^\alpha_{\text{based}}(\mathbb{P}^1, G/B)$, $\infty \mapsto *$ $\curvearrowleft$ $\mathbb{C}^* \times T = \mathbb{T}$

aka quasimap space

$\tilde{U}_\hbar(\mathfrak{g}^\vee)$ = Rees alg. of $U(\mathfrak{g}^\vee) \otimes_{\mathrm{Sym}(\mathfrak{g}^\vee)} \mathrm{Sym}(\mathfrak{h}^\vee)$ $\mathfrak{g}^\vee$: Langlands dual

$$\tilde{U}_\hbar(\mathfrak{g}^\vee) \longrightarrow \bigoplus_{\alpha,\beta} \mathrm{Ext}^*_{\mathbb{T}}(IC(X^\alpha), IC(X^\beta)) \qquad X^\alpha, X^\beta \longrightarrow X^{\alpha+\beta}$$

not isom. need a care.

3)' finite W-algebra of type A (Braverman-Feigin-Finkelberg-Rybnikov, N)

M^α = (generalized) Laumon space $\overline{\mathrm{Map}^\alpha(\mathbb{P}^1 \longrightarrow \text{partial flag})}$, $\infty \longmapsto *$, $\to X^\alpha$ = Zastava space

small resolution

3)'' affine version of 3) Braverman

4) [Braverman-Finkelberg-N]

$\mathcal{U}^d_G$ = Uhlenbeck space = $\overline{\mathrm{Bun}^d_G}$: moduli of G-bdles on $\mathbb{P}^2$ + framing at ℓ_∞

$W_A(\mathfrak{g})$ integral form of W-alg. "$\longrightarrow$" $\bigoplus \mathrm{Ext}^*(IC(\mathcal{U}^d_G), IC(\mathcal{U}^{d'}_G))$

We analyse simple modules of $\mathcal{A}$ via geometry of fixed pts in X :

center $\supset H_T^*(pt) = \mathbb{C}[\mathrm{Lie}\, T]$

Choose $\lambda \in \mathrm{Lie}\, T$ evaluation at $\lambda \Rightarrow \mathbb{C}[\mathrm{Lie}\, T] \to \mathbb{C}$

$\therefore$ simple modules are module of the specialized algebra

$$\mathcal{A} \otimes_{H_T^*(pt) \nearrow \lambda} \mathbb{C} = \mathrm{Ext}_T^*(\mathcal{L}, \mathcal{L}) \otimes_{H_T^*(pt)} \mathbb{C} = \mathrm{Ext}_A^*(\mathcal{L}, \mathcal{L}) \otimes_{H_A^*(pt)} \mathbb{C}$$

$A = \overline{\exp \mathbb{R}\lambda}$
$(A \subset T :$ subtorus $)$

We apply the localization thm to go to the fixed pt set, but not in a naive way. Go in two steps.

$$X \overset{j}{\hookleftarrow} X^+ \underset{p}{\overset{i}{\rightleftarrows}} X^A$$

$X^+ =$ attracting set $= \{ x \in X \mid \lim_{t \to 0} \lambda(t) x \text{ exists} \}$

hyperbolic restriction : • $\mathrm{Ext}_A^*(\mathcal{L}, \mathcal{L}) \otimes_{H_A^*(pt)} \mathbb{C} \cong \mathrm{Ext}_A^*(i^* j^! \mathcal{L}, i^* j^! \mathcal{L}) \otimes_{H_A^*(pt)} \mathbb{C}$
(Braden)

• $i^* j^! \mathcal{L}$ is semisimple cpx $\cong \bigoplus_w \mathcal{L}_w \otimes L(w)$

simple multiplicity

$\therefore \cong \bigoplus \mathrm{Ext}^*_{\text{,ey}}(\mathcal{L}_w, \mathcal{L}_{w'}) \otimes \mathrm{Hom}(L_w, L_{w'})$

$\Longrightarrow \{ L_w \}$: the set of simple modules !

we assume that nonconstant local system does not appear for simplicity

$\therefore$ $\mathcal{L}_w = IC(X_w^A)$ $\qquad$ $X_w^A \subset X^A$ locally closed smooth subvar.

Introduce a Verma-like module = standard module

Take $x_w \in X_w^A \Rightarrow M(w) :=$ ${}^!$stalk of $\underline{i^* j^! \mathcal{L}}$ at x_w is a module of $\mathcal{A} \otimes_{H_T^*(pt)} \mathbb{C}_\lambda$

$$\overset{\shortparallel}{\bigoplus_y IC(X_y^A) \otimes L(y)}$$

$\therefore$ multiplicity = dim ${}^!$stalk of $IC(X_y^A)$ at x_w

Remaining problem : Analyse X^A and its stratification.

§2. Fixed pts in Zastava

Recall $Z^\alpha = \overset{\alpha}{\mathrm{Map}}(\underset{\infty}{\mathbb{P}^1}, \underset{*}{G/B}) \curvearrowleft \mathbb{C}^* \times T = \mathbb{T}$

$(\hbar, \lambda) \in \mathrm{Lie}\,\mathbb{T}$ Assume $\hbar = 1$, $\lambda : \mathbb{C}^* \to T$ & regular dominant for simplicity

Prop. $\bigcup_\alpha (Z^\alpha)^A = \mathcal{B}_e$ s.t. stratum $= \mathcal{B}^w \cap \mathcal{B}_e$

↑ open cell

proof : $\varphi \in \mathrm{Map}^\alpha(\mathbb{P}^1, \mathcal{B})$, $\mathbb{C}^*$ acts on $\mathbb{P}^1$ and on $\mathcal{B}$ through λ fixed pt $=$ $\mathbb{C}^*$-equivariant map

∴ determined by $\varphi(1) \in \mathcal{B}$

$\varphi(\infty) = e \Rightarrow \varphi(1) \in \mathcal{B}_e$

$\varphi(0) \in \mathcal{B}^T$ ∴ $\varphi(0) \overset{\exists}{=} w \in W$ & $\varphi(1) \in \mathcal{B}^w$

$\deg \alpha \Longrightarrow \alpha = \lambda - w\lambda$ i.e. w is determined by α //

$\Longrightarrow$ KL conj. for $\mathfrak{g}$: finite